IS THE SEMI-CLASSICAL ANALYSIS VALID FOR EXTREME BLACK HOLES?

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Abstract

The surface gravity for the extreme Reissner-Nordström black hole is zero suggesting that it has a zero temperature. However, the direct evaluation of the Bogolubov's coefficients, using the standard semi-classical analysis, indicates that the temperature of the extreme black hole is ill definite: the Bogolubov's coefficients obtained by performing the usual analysis of a collapsing model of a thin shell, and employing the geometrical optical approximation, do not obey the normalization conditions. We argue that the failure of the employement of semi-classical analysis for the extreme black hole is due to the absence of orthonormal quantum modes in the vicinity of the event horizon in this particular case.

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1 Introduction

The possibility that a black hole may radiate with a planckian spectrum was first pointed out in the seminal paper of Hawking [1]. A collapsing model for the formation of the black hole was considered: a spherical mass distribution collapses under the action of gravity, leading to a final state where all mass is hidden behind an event horizon. The initial state is a Minkowski space-time. The final state is the Schwarzschild space-time which is asymptotically flat. Quantum fields are considered in this dynamical configuration. The main point is that the initial vacuum state does not coincide with the final vacuum state. From the point of view of an observer at the spatial infinity, after the formation of the black hole particles are created with a planckian spectrum. This fact allows to attribute

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to the black hole a temperature. For the Schwarzschild black hole, the temperature is $T = 1/(8\pi M)$, where M is the mass of the black hole.

The analysis of Hawking radiation for more general cases, like the Reissner-Nordström black hole, leads also to the notion of temperature due to the planckian form of the spectrum of the emitted particles. In general, the expression for the temperature for a black hole, using a semi-classical analysis (in the sense of propagation of quantum fields in the geometric optical approximation), is given by $T = \kappa/2\pi$ where κ is the surface gravity. The Hawking radiation is also well definite for rotating black holes, charged or not. A particular case occurs for the so-called extreme black holes, for which the surface gravity is zero. In this case, it is generally stated that the temperature is also zero. In fact, considering the Reissner-Nordström solution, the expression for the temperature is given by

$$T = \frac{1}{8\pi M}\left(1 - 16\pi^2\frac{Q^4}{A^2}\right) \quad , \tag{1}$$

where $A = 4\pi r_+^2$, $r_+ = M + \sqrt{M^2 - Q^2}$, Q is the charge of the black hole and M its mass. In the limit $Q = M$, $T = 0$.

However, this definition of the temperature of the extreme black hole as the limit of the temperature of the non-extreme black hole when $Q \rightarrow M$ may hide some subtle points about the thermodynamics of the extreme black hole. The main question we would like to address is the following: is it possible to obtain the zero temperature by performing a semi-classical analysis if the extreme condition is imposed from the beginning? The goal of this work is to show that such analysis contains many controversial aspects and it is very like that no semi-classical analysis is possible for the extreme black hole.

There has been many discussions on the real existence of an extreme black hole. Perturbative considerations based on the expansion of the energy-momentum tensor of quantum fields coupled to Einstein's equations led to doubts on the possibility to have extreme black hole solutions [2]. Moreover, it has been argued that the existence of a zero temperature black hole would violate the third law of thermodynamics unless the weak energy condition is not satisfied [3]. But, an analysis of the collapse of a charged thin shell indicates that, classically, an extreme black hole can be formed [4]. One assumption of the present paper, based on the results of reference [4], is that an extreme black hole can be formed through gravitational collapse.

Let us review the main steps of the evaluation of the temperature of a black hole sketched above. The semi-classical analysis of the thermodynamics of a black hole is generally performed considering the formation of the black hole due to the gravitational collapse of a spherical distribution of mass (see references [1, 8, 9, 10, 11]). Initially the mass density is almost zero, and the space-time is the Minkowski one. Later, the collapse of the mass distribution leads to the formation of an event horizon, which characterizes the black hole. The space-time after the formation of the black hole is asymptotically flat. The vacuum state of a quantum field before the collapse of the mass distribution to a black hole does not coincide with the vacuum state for the same quantum field after the appearance of the black hole. The computation of the Bogolubov's coefficients between the *in* and *out* quantum states leads to the notion of temperature. The Bogolubov's coefficients, which allows to express the quantum states *out* in terms of the quantum

states *in*, must obey some normalization conditions, which can also be seen as a set of compatibility conditions.

The application of the procedure described above to the extreme Reissner-Nordström black hole has been discussed in references [5, 6, 7]. In references [5, 6] it has been pointed out that the extreme black hole does not behave as thermal object. Moreover, the number of particle created for each frequency ω is infinite. The authors have exploited an analogy between the extreme black hole and the uniformly accelerated mirror. These conclusions have been criticized in reference [7] who argued that a modification in the calculations is needed in order to give sense to some mathematical steps. Moreover, the author of reference [7] has considered a wave packet instead of a simple plane wave expansion.

It is well known that the construction of a wave packet may eliminate divergent expressions when a pure plane wave expansion is considered. This possibility was already stressed in [5]. In the present paper we would like to point out that the problem of evaluation of the Bogolubov's coefficients for the extreme black holes is more delicate than stated in references [5, 6]. Not only the number of particles are infinite, but also the normalization conditions for the Bogolubov's coefficients are not obeyed. This is due essentially to the properties of the Bogolubov's coefficient $\alpha_{\omega\omega'}$: in the extreme case, the computation of the modulus of this coefficient leads to non-convergent integrals. This may indicate that the computation of Bogolubov's coefficients has no sense for the extreme case, at least in the framework of a semi-classical analysis.

The modification introduced by [7] does not change the situation. This modification consists in considering a logarithmic term which is present in the expression for the tortoise radial coordinate r^*. This logarithmic term is sub-dominant near the horizon. However, as it will be shown later, the logarithmic term is not necessary, since all mathematical expression has a sense when it is not considered. Even when it is taken into account, the problem remains essentially the same. Moreover, the construction of a wave packet can not change the result since it would consist, for the extreme case, in the superposition of modes that do not obey the normalization condition.

The reason for this curious result is not obvious. But, we will argue that the failure of semi-classical analysis lies on the causal structure of the space-time generated by an extreme black hole: the *in* and *out* states would not be connected due to the fact that near the event horizon the quantum modes do not admit an orthonormal basis. The loss of normalization of the quantum modes near the horizon is due to the fact that the near horizon geometry is a portion of the anti-deSitter space-time. It is important to remark that the existence of a zero temperature black hole would imply the violation of the third law of thermodynamics. From this point of view, the results reported in this paper may just re-state the validity of the third law of thermodynamics, at least in the context of the semi-classical analysis.

This paper is organized as follows. In next section, we review the computation of Hawking radiation for the Reissner-Nordström black holes. We use the simplified scenario of a thin charged collapsing shell, following the analysis presented in reference [8]. This allows us to fix notation and some important relations, as the normalization conditions for the Bogolubov's coefficients. In section 3, we redone this analysis for the extreme case, showing explicitly that the normalization conditions are not satisfied. In section 4, the

behaviour of quantum modes near the horizon is discussed. We present our conclusions in section 5.

2 Hawking radiation for a Reissner-Nordström Black Hole

The Reissner-Nordström solution for a static spherically symmetric space-time with a constant radial electric field is given by

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad , \tag{2}$$

where M is the mass and Q is the charge. This solution can be rewritten as

$$ds^2 = \left(1 - \frac{r_-}{r}\right)\left(1 - \frac{r_+}{r}\right) dt^2 - \left\{\left(1 - \frac{r_-}{r}\right)\left(1 - \frac{r_+}{r}\right)\right\}^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad , \tag{3}$$

where $r_\pm = M \pm \sqrt{M^2 - Q^2}$. Black hole solutions implies $M \geq Q$, while naked singularities appear if $M < Q$. The case $M = Q$ corresponds to the extreme black hole solution. This form of the metric leads to the new coordinates

$$u = t - r^* \quad , \quad v = t + r^* \quad , \tag{4}$$

$$r^* = r + \frac{r_+^2}{r_+ - r_-} \ln\left[\frac{r}{r_+} - 1\right] - \frac{r_-^2}{r_+ - r_-} \ln\left[\frac{r}{r_-} - 1\right] \quad . \tag{5}$$

The metric may be written in terms of these new coordinates as

$$ds^2 = \left(1 - \frac{r_-}{r}\right)\left(1 - \frac{r_+}{r}\right) du\, dv - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad . \tag{6}$$

From now on, a two-dimensional model will be considered, ignoring the angular terms. For the quantum fields to be considered later, this is equivalent to consider the angular term with $l = 0$ (zero angular momentum). But, the final results do not depend on this assumption, being valid for general l.

Let us consider the simplified model of a collapsing thin shell. The collapse of a charged thin shell has been studied in details in reference [4]. When $t \to -\infty$, the density of the shell goes to zero and the space-time is flat. Hence, at past infinity a scalar quantum field may be expanded into the normal modes

$$\phi = \int d\omega \frac{1}{\sqrt{4\pi\omega}} \left(a_\omega e^{-i\omega v} + a_\omega^\dagger e^{i\omega v}\right) \quad , \tag{7}$$

where we have just considered the incoming modes given by the coordinate v. After the collapse of the shell, a black hole is formed given a space-time described by the metric (3), which is asymptotically flat. Hence the outcome mode at $t \to \infty$ is given by

$$\phi = \int d\omega \frac{1}{\sqrt{4\pi\omega}} \left(b_\omega e^{-i\omega u} + b_\omega^\dagger e^{i\omega u}\right) \quad . \tag{8}$$

The problem to solve is how to connect the coordinates u and v, obtaining in this way the Bogolubov's coefficients of the transformation

$$\frac{e^{-i\omega u}}{\sqrt{4\pi\omega}} = \int_0^\infty \left\{ \alpha_{\omega\omega'} e^{-i\omega' v} + \beta_{\omega\omega'} e^{i\omega' v} \right\} \frac{d\omega'}{\sqrt{4\pi\omega'}} \tag{9}$$

with the inverse transformation

$$\frac{e^{-i\omega v}}{\sqrt{4\pi\omega}} = \int_0^\infty \left\{ \alpha^*_{\omega'\omega} e^{-i\omega' u} - \beta_{\omega'\omega} e^{i\omega' u} \right\} \frac{d\omega'}{\sqrt{4\pi\omega'}} \quad . \tag{10}$$

The coefficient $\beta_{\omega\omega'}$ is connected with the number of particles detected by an observer in the future infinity for each frequency ω. The Bogolubov's coefficients satisfy the consistency relations,

$$\int_0^\infty \left\{ \alpha_{\omega\omega'} \alpha^*_{\omega''\omega'} - \beta_{\omega\omega'} \beta^*_{\omega''\omega'} \right\} d\omega' \;=\; \delta(\omega - \omega'') \quad , \tag{11}$$

$$\int_0^\infty \left\{ \alpha_{\omega\omega'} \beta_{\omega''\omega'} - \beta_{\omega\omega'} \alpha_{\omega''\omega'} \right\} d\omega' \;=\; 0 \quad . \tag{12}$$

Notice that

$$\int_0^\infty \int_0^\infty \left\{ \alpha_{\omega\omega'} \alpha^*_{\omega''\omega'} - \beta_{\omega\omega'} \beta^*_{\omega''\omega'} \right\} d\omega' d\omega'' = 1 \quad , \tag{13}$$

a normalization condition to be used later.

An incoming mode v comes from the infinity past, traversing the collapsing shell, becoming later an outcome mode, traversing again the shell and attaining the infinity future. The modes are continuous, in the sense that $v|_{R=R_1} = V|_{R=R_1}$, $V|_{R=0} = U|_{R=0}$, $U|_{R=R_2} = u|_{R=R_2}$, where u and v are the outgoing and incoming modes in the external geometry determined by the shell, while U and V are the same modes in the internal, minkowskian geometry, and R_1 and R_2 are the radius of the shell at the first and second crossing, respectively. An important point in this derivation is that the collapse is accelerated in such a way that the speed of the collapsing shell approach the speed of the light at the moment when the event horizon is formed.

At the second crossing, which we admit to occur near the moment of formation of the black hole, the continuity of the metric leads to

$$dT^2 - dR^2 = \left[1 - \frac{r_+}{R}\right]\left[1 - \frac{r_-}{R}\right] dt^2 - \left\{ \left[1 - \frac{r_+}{R}\right]\left[1 - \frac{r_-}{R}\right] \right\}^{-1} dR^2 \quad . \tag{14}$$

At the moment of the second crossing, we may consider that

$$R \approx r_+ + A(T_0 - T) \quad , \tag{15}$$

where A is a constant and T_0 is the time where the black hole is formed. Inserting this relation in (14), and considering the continuity of the incoming modes, what allows to express T in terms of v, it results the following relation between the u and v modes:

$$u = -2\sigma \ln\left[\frac{v_0 - v}{C}\right] \tag{16}$$

where $\sigma = \kappa^{-1} = r_+^2/(r_+ - r_-)$ is the inverse of the surface gravity, C is a constant and $v_0 = T_0 - r_+$. Using the inner product for complex scalar fields,

$$(\phi_1, \phi_2) = -i \int d\Sigma^\mu \Big(\phi_1 \partial_\mu \phi_2^* - \partial_\mu \phi_1 \phi_2^* \Big) = -i \int d\Sigma^\mu \phi_1 \overleftrightarrow{\partial}_\mu \phi_2^* \quad , \tag{17}$$

and defining $f_\omega = e^{-i\omega v}/\sqrt{4\pi\omega}$, $g_\omega = e^{-i\omega u(v)}/\sqrt{4\pi\omega}$, we obtain the following expressions for the Bogolubov coefficients:

$$\alpha_{\omega\omega'} = (g_\omega, f_{\omega'}) \quad , \quad \beta_{\omega\omega'} = -(g_\omega, f^*_{\omega'}) \quad . \tag{18}$$

Due to the relation between u and v (16), the Bogolubov coefficients can be expressed in terms of the integrals

$$\alpha_{\omega\omega'} = \frac{1}{4\pi\sqrt{\omega\omega'}} \exp[i(\omega' v_0 - 2\sigma\omega \ln C - 2\sigma\omega \ln \omega')] \int_0^\infty e^{-i(y - 2\sigma\omega \ln y)} \Big\{ 1 + \frac{2\sigma\omega}{y} \Big\} dy \quad , \tag{19}$$

$$\beta_{\omega\omega'} = \frac{1}{4\pi\sqrt{\omega\omega'}} \exp[-i(\omega' v_0 + 2\sigma\omega \ln C + 2\sigma\omega \ln \omega')] \int_0^\infty e^{i(y + 2\sigma\omega \ln y)} \Big\{ 1 - \frac{2\sigma\omega}{y} \Big\} dy \quad , \tag{20}$$

where $y = \omega'(v_0 - v)$. Making an integration by parts and using the integral relation [1]

$$\int_0^\infty e^{\pm iy} y^{ia} dy = \frac{\Gamma[1 + ia]}{(\pm i)^{\mp i\pi/2 + ia}} = \mp a \Gamma(ia) e^{\mp\pi/2}, \tag{21}$$

it results the following expressions for the Bogolubov's coefficients:

$$\alpha_{\omega\omega'} = \frac{2\sigma\omega}{2\pi\sqrt{\omega\omega'}} \Gamma(2i\sigma\omega) e^{\sigma\pi\omega} \exp[i(\omega' v_0 - 2\sigma\omega \ln C - 2\sigma\omega \ln \omega')] \quad , \tag{22}$$

$$\beta_{\omega\omega'} = -\frac{2\sigma\omega}{2\pi\sqrt{\omega\omega'}} \Gamma(2i\sigma\omega) e^{-\sigma\pi\omega} \exp[-i(\omega' v_0 + 2\sigma\omega \ln C + 2\sigma\omega \ln \omega')] \quad . \tag{23}$$

It is important to stress that imposing the extreme condition $\sigma \to \infty$, the β coefficient becomes zero.

Before evaluating the Hawking temperature, we must notice that the above solutions for the Bogolubov's coefficients satisfies the consistency relations (11,12). It will be shown only the relation (11) rewritten as in (13). Computing the first term in the left hand side of (13), it comes out,

$$\begin{aligned}
&\int_0^\infty \int_0^\infty d\omega' d\omega'' \alpha_{\omega\omega'} \alpha^*_{\omega''\omega'} \\
&= \frac{\sigma^2}{\pi^2} \int_0^\infty \int_0^\infty \frac{d\omega'}{\omega'} d\omega'' \sqrt{\omega\omega''} \Gamma(2i\sigma\omega) \Gamma(-2i\sigma\omega'') e^{\pi\sigma(\omega + \omega'')} e^{\{-2i\sigma(\omega - \omega'')[\ln C + \ln \omega']\}} \quad , \\
&= \frac{2\sigma^2}{\pi} \int_0^\infty d\omega'' \sqrt{\omega\omega''} \Gamma(2i\sigma\omega) \Gamma(-2i\sigma\omega'') e^{\pi\sigma(\omega + \omega'')} e^{[-2i\sigma(\omega - \omega'')]} \delta[2\sigma(\omega - \omega'')] \quad , \\
&= \frac{1}{2} \frac{e^{2\pi\sigma\omega}}{\sinh(2\pi\sigma\omega)} \quad ,
\end{aligned} \tag{24}$$

[1] It could be argued that this expression is not definite strictly speaking when the exponential is pure imaginary. However, the limit case of a pure imaginary exponential term is well definite.

where we have used the relation

$$\Gamma(ia)\Gamma(-ia) = \frac{\pi}{a\sinh(\pi a)} \quad . \tag{25}$$

A similar calculation leads to

$$\int_0^\infty \int_0^\infty d\omega' d\omega'' \beta_{\omega\omega'} \beta^*_{\omega''\omega'} = \frac{1}{2} \frac{e^{-2\pi\sigma\omega}}{\sinh(2\pi\sigma\omega)} \quad . \tag{26}$$

From these expressions, the normalization condition (13) can be easily obtained:

$$\begin{aligned} \int_0^\infty \int_0^\infty d\omega' d\omega'' \alpha_{\omega\omega'} \alpha^*_{\omega''\omega'} \quad &- \quad \int_0^\infty \int_0^\infty d\omega' d\omega'' \beta_{\omega\omega'} \beta^*_{\omega''\omega'} = \\ &= \quad \frac{1}{2} \frac{e^{2\pi\sigma\omega}}{\sinh(2\pi\sigma\omega)} - \frac{1}{2} \frac{e^{-2\pi\sigma\omega}}{\sinh(2\pi\sigma\omega)} = 1 \quad . \end{aligned} \tag{27}$$

The Hawking temperature can be obtained in two equivalent ways. First by computing the number of particles with frequency ω in the future infinity,

$$N_\omega = \int_0^\infty \int_0^\infty d\omega' d\omega'' \beta_{\omega\omega'} \beta^*_{\omega''\omega'} \tag{28}$$

or by noticing that

$$||\alpha_{\omega\omega'}|| = e^{\pi\sigma\omega} ||\beta_{\omega\omega'}|| \tag{29}$$

and using the normalization condition (13). In both cases the result is

$$N_\omega = \frac{1}{e^{2\pi\sigma\omega} - 1} \quad . \tag{30}$$

This is characteristic of a Planckian spectrum with temperature $T = 1/(2\pi\sigma)$. In the non-extreme Reissner-Nordström case treated before, this temperature reads

$$T = \frac{1}{8\pi M}\left(1 - \frac{16\pi^2 Q^4}{A^2}\right) \quad , \tag{31}$$

where $A = 4\pi r_+^2$ is the area of the event horizon. It can be verified that when $Q \to M$, $T \to 0$.

3 The extreme black hole

The extreme condition $M = Q$, leads to the metric

$$ds^2 = \left\{1 - \frac{M}{r}\right\}^2 dt^2 - \left\{1 - \frac{M}{r}\right\}^{-2} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad . \tag{32}$$

The null coordinates take now the form

$$u = t - r^* \quad , \quad v = t + r^* \quad , \tag{33}$$

$$r^* = r + 2M\ln\left(\frac{r}{M} - 1\right) - \frac{M}{r/M - 1} \quad , \tag{34}$$

The extreme black hole has a degenerate event horizon with $r_+ = r_-$. The new tortoise coordinate (34) is quite different from the preceding one for the non-extreme case, (5), mainly due to the last term in (34). But, as it can be verified, expression (34) may be obtained as a limit case of (5) when $r_- \to r_+$.

The same model of the preceding section will be considered now on: a collapsing thin shell, with the space-time external to the shell being determined by the metric (32), while the internal space-time is minkowskian. At the past infinity, all space-time is essentially minkowskian and a quantum scalar field admits the decomposition (7), while in the future infinity the space-time is asymptotically minkowskian and the quantum scalar field admits the decomposition (8). The task now is the same as before: to connect both quantum states.

In this sense, an ingoing mode, described by the null coordinate v comes from the past infinity, traverses the thin shell when space-time is essentially Minkowski, becoming later an outgoing mode, which traverses again the shell near the moment of the formation of the black hole. The same match conditions established before can be used. Repeating all the calculations performed before we find now

$$u = \frac{C}{v_0 - v} \quad , \tag{35}$$

where, as before, C and v_0 are constants connected with some parameters characterizing the collapse of the shell. In performing this evaluation, we considered just the leading term near the horizon in (34). The problem of taking into account the sub-dominant logarithmic term will be discussed later. But, we may already state that taking into account this sub-dominant term does not change the essential of the results.

The expressions (18) are used again to compute the Bogolubov coefficients, leading to

$$\begin{aligned}
\alpha_{\omega\omega'} &= (g_\omega, f_{\omega'}) = \frac{1}{4\pi\sqrt{\omega\omega'}}\int_{-\infty}^{v_0} e^{-i\frac{\omega C}{v_0 - v} + i\omega' v}\Big\{\omega' + \frac{\omega C}{(v_0 - v)^2}\Big\}dv \\
&= \frac{e^{i\omega' v_0}}{4\pi\sqrt{\omega\omega'}}\int_0^\infty e^{-i(y + D/y)}\Big\{1 + \frac{D}{y^2}\Big\}dy \quad ; \qquad (36) \\
\beta_{\omega\omega'} &= -(g_\omega, f^*_{\omega'}) = \frac{1}{4\pi\sqrt{\omega\omega'}}\int_{-\infty}^{v_0} e^{-i\frac{\omega C}{v_0 - v} - i\omega' v}\Big\{\omega' - \frac{\omega C}{(v_0 - v)^2}\Big\}dv \\
&= \frac{e^{-i\omega' v_0}}{4\pi\sqrt{\omega\omega'}}\int_0^\infty e^{i(y - D/y)}\Big\{1 - \frac{D}{y^2}\Big\}dy \quad , \qquad (37)
\end{aligned}$$

with $y = \omega'(v_0 - v)$. With the aid of the variable redefinition $y = \pm D/z$, $D = C\omega'\omega$, the expression for the Bogolubov's coefficients take the form,

$$\begin{aligned}
\alpha_{\omega\omega'} &= \frac{1}{2\pi}\frac{e^{i\omega' v_0}}{\sqrt{\omega\omega'}}D\int_0^\infty e^{-i(y + D/y)}\frac{dy}{y^2} \quad , \qquad (38) \\
\beta_{\omega\omega'} &= \frac{1}{4\pi}\frac{e^{-i\omega' v_0}}{\sqrt{\omega\omega'}}D\Big\{\int_{-\infty}^0 e^{i(y - D/y)}\frac{dy}{y^2} - \int_0^\infty e^{i(y - D/y)}\frac{dy}{y^2}\Big\} \quad . \qquad (39)
\end{aligned}$$

These integrals may be solved by using the integral representation of modified Bessel functions of second kind,

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{e^{-t-\frac{z^2}{4t}}}{t^{\nu+1}} dt \quad . \tag{40}$$

Using the expressions for the Bogolubov's coefficients (38,39) and this integral representation, it results [2]

$$\alpha_{\omega\omega'} = \pm\frac{e^{i\omega' v_0}}{\pi}\sqrt{\frac{D}{\omega\omega'}}K_1(\pm 2i\sqrt{D}) = \mp\frac{e^{i\omega' v_0}}{\pi}\sqrt{\frac{D}{\omega\omega'}}H_1^{(1)}(\mp 2\sqrt{D}) \quad , \tag{41}$$

$$\beta_{\omega\omega'} = \pm i\frac{e^{-i\omega' v_0}}{\pi}\sqrt{\frac{D}{\omega\omega'}}K_1(\pm 2\sqrt{D}) \quad , \tag{42}$$

where $H_1^{(1)}(x)$ is the Hankel's function of first kind. These expressions are very similar to the corresponding ones for the uniformly accelerated mirror [10]. Aside some unimportant differences in constant factors, there is an additional phase in the accelerated mirror problem, what can distinguish crucially the mirror problem from the extreme black hole problem.

The first thing to remark about these expressions for the Bogolubov coefficients, it is the absence of a simple relation between them, in contrast of what happens with the non-extreme case. This does not allow to extract the notion of a temperature for the extreme case. This induces to state that the extreme black hole is not a thermal object. However, the situation is more complex yet. In fact, the Bogolubov coefficients found before do not satisfy the normalization condition (13). Let us verify now this, calculating separately the two terms of (13).

Let us first compute the second term of (13). Using the solution found before, the second term of (13) may be written as

$$\int_0^\infty\int_0^\infty d\omega' d\omega'' \beta_{\omega\omega'}\beta^*_{\omega''\omega'} = \frac{C}{\pi^2}\int_0^\infty\int_0^\infty d\omega' d\omega'' K_1(2\sqrt{C\omega\omega'})K_1(2\sqrt{C\omega''\omega'}) \quad . \tag{43}$$

The integral in ω'' may be easily computed. In fact, it takes the form

$$\int_0^\infty d\omega'' K_1(2\sqrt{C\omega''\omega'}) = \frac{1}{2C\omega'}\int_0^\infty dy\, y\, K_1(y) \tag{44}$$

where $y = 2\sqrt{C\omega'\omega''}$. In order to evaluate this integral, we use the following integral representation of the modified Bessel function of second kind:

$$K_1(z) = \int_0^\infty e^{-z\cosh\theta}\cosh\theta\, d\theta \quad . \tag{45}$$

[2]Notice that the integral representation (40) is valid for $|\arg z| < \pi/2$. The computation of $\alpha_{\omega\omega'}$ implies $|\arg z| = \pi/2$. This limit case is, however, well definite. This can be seen by expanding the exponential in terms of sinus and cosinus and using the integrals $3.868, 1-4$ of the Gradstein and Ryzhik table [12] after differentiating them with respect to one of the parameters.

Hence,

$$\begin{aligned}\int_0^\infty K_1(y)y\,dy &= \int_0^\infty\int_0^\infty e^{-y\cosh\theta}y\,\cosh\theta\,dyd\theta \\ &= \int_0^\infty\int_0^\infty e^{-x}x\,dx\,\frac{d\theta}{\cosh\theta} = \int_0^\infty \frac{d\theta}{\cosh\theta} = \frac{\pi}{2} \quad ,\end{aligned} \tag{46}$$

where we made the substitution $x = y\cosh\theta$. In this way, the second term of (13) can be written as

$$\begin{aligned}\int_0^\infty\int_0^\infty d\omega' d\omega'' \beta_{\omega\omega'}\beta^*_{\omega''\omega'} &= \frac{1}{4\pi}\int_0^\infty K_1(2\sqrt{C\omega\omega'})\,\frac{d\omega'}{\omega'} \\ &= -\frac{1}{2\pi}\int_0^\infty K_1(x)\,\frac{dx}{x} \\ &= -\frac{1}{2\pi}\int_0^\infty\int_0^\infty e^{-x\cosh\theta}\cosh\theta\,d\theta\,\frac{dx}{x} \quad .\end{aligned} \tag{47}$$

where the integral representation of the function $K_1(x)$ (45) has been used again. Under the substitution $u = x\cosh\theta$, this term takes the form

$$\int_0^\infty\int_0^\infty d\omega' d\omega'' \beta_{\omega\omega'}\beta^*_{\omega''\omega'} = \frac{1}{2\pi}\int_0^\infty \cosh\theta\,d\theta\int_0^\infty e^{-u}\,\frac{du}{u} \tag{48}$$

which is obviously a divergent term.

Until this moment, we may keep the hope to recover the normalization condition, since this divergent term may be cancelled by another divergent term coming from the first term in (13). Let us now compute this term, choosing the minus sign in (41). It is more convenient to use the representation in terms of Hankel's function in the expression for $\alpha_{\omega\omega'}$. Defining $x = 2\sqrt{C\omega''\omega'}$ and $y = 2\sqrt{C\omega\omega'}$, the final expression for the first term in (13) takes the form

$$\int_0^\infty d\omega' d\omega'' \alpha_{\omega\omega'}\alpha^*_{\omega''\omega'} = \frac{1}{4}\int_0^\infty H_1^{(2)}(x)x\,dx\int_0^\infty H_1^{(1)}(y)\,\frac{dy}{y} \quad , \tag{49}$$

where we have used the fact that, for real values of the argument ${H^{(1)}}^*_1(x) = H_1^{(2)}(x)$, $H_1^{(2)}(x)$ being the Hankel's functions of the second kind. The second integral in the right hand side of (49) is a divergent term; however, the first integral is not convergent, as an asymptotic analysis indicates. In fact, for large values of the argument, $H_1^{(1,2)}(x) \rightarrow \sqrt{\frac{2}{\pi x}}e^{\pm(x-\frac{\pi}{2}-\frac{\pi}{4})}$ and the integrand oscillates with increasing amplitude. The order the integrations are performed does not change this result

With the results obtained above, it is easy to see that the normalization condition is not satisfied. The second term in (13) is infinite. However, the first term contains non-convergent integrals; hence, its value is not definite. The conclusion is that the Bogolubov transformation between the *in* and *out* vacuum modes is ill definite for the extreme black hole, and no thermodynamics can be constructed with a collapse scenario.

We would like to stress that taking into account the logarithmic term in (34) does not change the scenario described before. To take into account this term means to consider

an expansion until second order. In fact, it can be shown that it is equivalent to express the collapse of the shell near the formation of the horizon as

$$R \approx M + A(T_0 - T) + B(T_0 - T)^2 \quad , \tag{50}$$

where B is another positive parameter. This form for the collapse is still consistent with the general analysis performed in reference [4]. All the expansions must now be carried out until second order, and the final results for the Bogolubov's coefficients are now expressed as a sum of modified Bessel functions. However, the problems pointed out above remain the same.

4 Quantum fields near the horizon

If a complete basis of orthonormal modes is implemented in the past infinity, the propagation of the modes to the future infinity should in principle be well definite. The fact that the normalization conditions are not satisfied when the Bogolubov's coefficients are evaluated in the future infinity reveals that an anomaly occurs in the propagation of quantum fields. Normally, the propagation of quantum fields is such that in any hypersurface at constant time a complete basis can be definite. That is what occurs in the non-extreme case. If the extreme case is the limit of the non-extreme one when $Q \to M$ we could expect the same normal behaviour in the propagation of quantum fields.

One important point is that the extreme RN black hole is not the limit of the non-extreme one. This has already been remarked in reference [13]. In this reference, the entropy law for the extreme black hole has been studied, and the authors have concluded that $S = 0$, in spite of the fact that the horizon area of the extreme black hole is non-zero. Hence, a violation of the black hole entropy law $S = A/4$ occurs for the extreme RN black hole. This results has been confirmed by the computation done in references [14, 15]. However, it has been argued that considering the string framework the usual entropy law could be recovered [16]. This controversy shows that the thermodynamics of extreme black holes is far from being a trivial subject.

In reference [13] the problem of computation of the temperature of extreme black hole has been addressed by employing the method of euclideanization of the metric. The conclusion was that the periodicity of the euclidean time is arbitrary for the extreme black hole; hence the temperature is also arbitrary and in any case non-zero. This result contrasts with the same evaluation made for the non-extreme case, where a precise temperature can be obtained through the periodicity of the euclidean time; moreover, the so-obtained temperature agrees with the results obtained by using the surface gravity and by using the Bogolubov's coefficients. This fact points again to the specificity of the extreme black hole, indicating that it is not the limit case of the non-extreme black holes. In reference [13] this specificity is connected with an unusual feature of the extreme black hole: as it can be explicitly verified from the metric (32), the spatial distance of any point to the event horizon is infinite; for the non-extreme case, this spatial distance is finite.

Another specific feature of the extreme black hole with respect to the non-extreme one concerns the geometry near the event horizon. This specificity has a close connection

with the spatial distance to the event horizon quoted before. For the non-extreme black hole, the metric near the horizon takes the form

$$ds^2 \approx \frac{r_+ - r_-}{r_+^2}\rho dt^2 - \frac{r_+^2}{r_+ - r_-}\frac{d\rho^2}{\rho} - r_+^2 d\Omega = \rho' dt'^2 - \frac{d\rho'^2}{\rho'} - r_+^2 d\Omega^2 \quad , \tag{51}$$

where

$$\rho = r - r_+ \quad , \quad \rho' = \frac{r_+^2}{r_+ - r_-}\rho \quad , \quad t' = \frac{r_+ - r_-}{r_+^2} t \quad . \tag{52}$$

For the extreme black hole, however, the same computation leads to

$$ds^2 \approx \rho^2 dt^2 - \frac{d\rho^2}{\rho^2} - M^2 d\Omega^2 \quad . \tag{53}$$

First of all, it must be remarked that the metric (53) is not the limit of (51) when $r_- \to r_+$. Notice that the extreme limit and the near horizon limit when applied for the metric (3) do not commute. The specificity of the extreme black hole with respect to the non-extreme ones comes from the geometry near the horizon. Far from the horizon, the extreme space-time can be obtained from the non-extreme one.

The geometry described by (53) corresponds to an anti-deSitter space-time. More precisely, to a portion of the anti-deSitter space-time. It is well known that the anti-deSitter space-time has special problems concerning the propagation of initial data defined on a given hyspersurface. In fact, the values of a given set of fields on a given hypersurface can be obtained from the initial data of these fields on another hypersurface only in a portion of the entire space-time. This is due to the fact that the anti-deSitter space-time has a timelike infinity or, in other words, this space-time is not globally hyperbolic. For a review of the properties of the anti-deSitter space-time, see [17, 18, 19]. The formulation of a quantum field theory in an anti-deSitter space-time has been studied in reference [20]. In this work, it has been shown that no Hilbert space can be implemented in such space-time, unless specific boundary conditions are fixed at infinity. To do so, it is necessary to use a universal covering of the anti-deSitter space-time. However, the geometry described by the metric (53) does not correspond to this universal covering.

From this considerations, we must expect that the problems with the normalization of the Bogolubov's coefficients must come from the impossibility of assign a Hilbert space for quantum fields in the space-time described by (53). In fact, let us solve the Klein-Gordon equation for the space-time described first by (51) and later by (53). The quantum modes can be obtained from the classical solutions with the canonical methods.

Using (51), the Klein-Gordon equation for a massless scalar field reduces to

$$\Box\phi = -\rho\phi'' - \phi' - \frac{\omega^2}{\rho} + \frac{l(l+1)}{r_+^2}\phi = 0 \tag{54}$$

where primes denote derivatives with respect to ρ, l is the angular momentum eigenvalue and ω is normal mode frequency. Solving this equation, we obtain

$$\phi = c_1 K_{i4\omega}\Big(\frac{\sqrt{l(l+1)}}{r_+}\sqrt{\rho}\Big)e^{i\omega t} \quad , \tag{55}$$

where K stands for the modified Bessel's function. This is the well-known basis of orthonormal modes of a massless scalar field for RN and Schwarzschild black holes (see [21] and references therein).

For the extreme case, the employement of the (53) reduces the Klein-Gordon equation to

$$-\rho^2\phi'' - 2\rho\phi' - \frac{\omega^2}{\rho^2} + l(l+1)\phi = 0 \quad , \tag{56}$$

with the solution

$$\phi = \frac{1}{\sqrt{\rho}}\Big[c_1 J_{l+1/2}\Big(\frac{\omega}{\rho}\Big) + c_2 J_{-(l+1/2)}\Big(\frac{\omega}{\rho}\Big)\Big]e^{i\omega t} \quad . \tag{57}$$

Now, if we compute the norm of these modes, in the Klein-Gordon sense, we obtain

$$\begin{aligned}(\phi,\phi) &= -i\int d\Sigma^\mu \phi \overleftrightarrow{\partial}_\mu \phi^* = 2\omega|c_{1,2}|^2\int_0^\infty \Big[J_{\pm(l+1/2)}\Big(\frac{\omega}{\rho}\Big)\Big]^2\frac{d\rho}{\rho^3} \\ &= 2\omega|c_{1,2}|^2\int_0^\infty x\Big[J_{\pm(l+1/2)}(\omega x)\Big]^2 dx \end{aligned}\tag{58}$$

where $x = 1/\rho$. It is easy to verify that the norm of the scalar modes for the extreme case near the horizon is divergent. In principle, this problem could be circumvented. Plane wave solutions in the Minkowski space-time are also divergent when integrated in all space. However, this difficulty is solved by defining the plane wave modes in a finite volume; at the end, when the physical quantities are evaluated, the limit of an infinite volume may be applied. Here, such "normalization" procedure can not be implemented due to one particularity of the space-time already pointed out: the spatial distance of any point to the event horizon is infinite; hence, any volume around the horizon is infinite. It is not possible to normalize the modes found above.

The conclusion is that it is not possible to assign a complete basis of orthonormal modes near the horizon for the extreme RN black hole. In this sense, we can understand the negative result of the preceding section: the global propagation of quantum fields from past infinity to future infinity is ill definite due to the anomalous behaviour of these quantum fields near the horizon. The modes cross the shell just before the horizon formation will propagate in the geometry described before. They are, in this situation, anymore normal modes. Hence, the basis at the future infinity is not anymore a complete basis of normal modes.

It is instructive to compare this situation with a similar one that occurs with AdS black holes [22, 23], which plays a crucial rôle in the AdS/CFT correspondence. In this case, the metric takes the form (53) at the spatial infinity, corresponding to $\rho \to \infty$, and not at $\rho \to 0$ as in the present case. Again, the solution of the Klein-Gordon equation takes the form (57). Now, one of the modes is divergent in the sense of the Klein-Gordon inner product, while the other one is finite, since the solutions are valid in the limit $x \to 0$ in expression (58). The choice of appropriate boundary conditions allows now to select just the normalizable modes. These features have already been discussed in references [22, 19]. Notice that in this AdS black hole case, the limit $\rho \to \infty$ corresponds to the

timelike boundary (where, following [20], the choice of convenient boundary conditions allows to give sense to the Hilbert space in an anti-deSitter space-time), in contrast what happens with the near horizon limit of the extreme black hole. Again in contrast with the extreme black hole case, in the AdS black hole the spatial distance of any point to the horizon is finite. Finally, it is important to remark that in the extreme black hole geometry, the other boundary corresponds to a Minkowski space-time, where we find plane wave modes; hence, there is apparently no way to select appropriate boundary conditions in order to avoid divergent quantum modes.

5 Conclusions

The surface gravity of the extreme Reissner-Nordström black hole is zero. For this reason, the temperature of the extreme black hole is believed to be zero. Moreover, if the temperature for the general Reissner-Nordström is evaluated and the extreme condition $M = Q$ is imposed in the final expression, it results $T = 0$. But, the notion of temperature must be extracted, for example, from the direct computation of the relation between the *in* and *out* through the Bogolubov's coefficients. In this paper, this evaluation of the Bogolubov's coefficients was performed by imposing from the beginning the extreme condition. We have found that the Bogolubov's coefficients do not satisfy the normalization condition (13) due, mainly, to the presence of non convergent integrals in the computation of the modulus of the coefficient $\alpha_{\omega\omega'}$. Hence, this semi-classical analysis seems to be ill definite for the extreme black hole.

This is a quite curious result. In general it is believed that the normalization condition must be satisfied by construction. If the normalization condition is not satisfied, it means that the construction itself is ill definite. What is the reason for the failure of this construction for the extreme black hole? A very important point to be noticed concerns the fact that, in spite of what we could think, the extreme RN black hole is not the limit of the non-extreme ones. This has already been stressed in reference [13]. This is due to the behaviour of the geometry near horizon. In the extreme case, in contrast of what happens in the non-extreme situation, the near horizon geometry is described by a portion of the anti-deSitter space-time.

The formulation of quantum field theory in an anti-deSitter space-time is not a trivial problem. In reference [20] it has been shown that a Hilbert space can be implemented in an anti-deSitter space-time if specific boundary conditions are fixed and if the universal covering of the anti-deSitter space-time is used. This is due to fact that the anti-deSitter space-time is not globally hyperbolic and its infinity is timelike. However, the near horizon geometry for extreme black hole is just a portion of the anti-deSitter space-time. Computing the normal modes in this case leads to non-normalized states. In principle, it is not possible in this case to recover the notion of Hilbert space, due to the absence of a universal covering of anti-deSitter space-time.

Following the computations for the extreme and non-extreme RN black holes we can easily see that the results obtained in the general case leads, after imposing the limit case $\kappa = 0$, to the expressions that are consistent with a zero temperature black hole: the $\beta_{\omega\omega'}$

coefficient goes to zero, but the $\alpha_{\omega\omega'}$ coefficient remains non null. On the other hand, the limit case $\kappa = 0$ when applied to the results obtained for the general case does not lead to the corresponding results obtained by imposing the extreme condition from the begining. Tracing back where the discrepancy begins to occur, we may verify that both cases loose contact at very begining, in the relations between u and v modes (16) and (35), for the general and extreme case, respectively. Notice that the tortoise coordinate (34) may be obtained from (5) in the extreme limit case. Again, the specificity of the relation between the *in* and *out* modes in the extreme case is due to matching condition imposed between them at the moment of the horizon formation. This stress again the crucial role played by the geometry near the event horizon.

In general lines, the results reported in this paper confirm, as far as the standard semi-classical analysis is concerned, those of references [5, 6] in stating that the extreme black hole does not behave as thermal object. But, it indicates also that, in principle, no semi-classical analysis at all can be consistently performed for the extreme black hole. This is also in agreement with the results of reference [13], stating the specificity of the extreme black hole and its failure to obey the general black holes's thermodynamics laws. Remark that the extreme black holes would be an example of violation of the third law of thermodynamics.

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